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ON THE STRUCTURE OF PRIMARY ABELIAN GROUPS OF COUNTABLE ULM TYPE

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Abstract. In this paper we will give structure theorems for abelian p-groups of countable Ulm type utilizing the notion of high subgroup introduced by John M. Irwin and its generalization, N-high subgroup, introduced by Irwin and E. A. Walker. The general technique employed is to give conditions under which automorphisms of these subgroups extend to automorphisms of the group.

In this paper we will give structure theorems for abelian p-groups of countable Ulm type utilizing the notion of high subgroup introduced by John M. Irwin and its generalization, N-high subgroup, introduced by Irwin and E. A. Walker. The general technique employed is to give conditions under which automorphisms of these subgroups extend to automorphisms of the group.

Let G be a reduced abelian p-group and β a limit ordinal. A chain of subgroups $\{G_n\}_{n\in\omega}$ of G is said to be a β-high chain in G if G_n is a $p^{\beta}G$-high subgroup of G. If σ is the natural homomorphism from G onto $G/p^{\beta}G$, then $\{\sigma(G_n)\}_{n\in\omega}$ is said to be a β-high chain in $G/p^{\beta}G$. We show that if G and H are reduced abelian p-groups such that $p^{\beta}G\cong p^{\beta}H$, $p^{\beta}G$ is torsion complete and there exists an isomorphism between $G/p^{\beta}G$ and $H/p^{\beta}H$ preserving the β-high chains, then G and H are isomorphic. In the case that $p^{\beta}G$ is not torsion complete, heavy restrictions must be imposed on these isomorphisms. In particular we have that a p-group G for which $G/p^{\omega}G$ is torsion complete and $p^{\omega}G$ is homogeneous with order bound p^n is completely determined by $G/p^{\omega}G$, $p^{\omega}G$ and its class of high subgroups.

To consider groups of Ulm type $\leqq\omega$, we introduce the idea of an ω-summable group: a group C is said to be *ω-summable* if $C=\bigcup_{n\in\omega} C_n$ such that $\{C_n\}_{n\in\omega}$ is a chain of $p^{\omega n}C$-high subgroups of C. We give necessary and sufficient conditions for two ω-summable groups to be isomorphic, and show that any abelian p-group of Ulm type $\leqq\omega$ is an isotype subgroup between an ω-summable group C and its completion with respect to the topology induced by $\{p^{\omega n}C\}_{n\in\omega}$.

For groups of countable Ulm type, we get similar results using "generalized" Ulm factors (for which the groups of Ulm type ω are a special case). From this we not only show the existence of reduced abelian p-groups of countable Ulm type

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having prescribed Ulm factors but obtain all such groups. This generalizes results of Fuchs and Kulikov as prescribed in Theorems 35.5 and 38.2 of [4].

Examples are then given to show that some of the structure theorems are the best possible, and to give greater insight as to why two reduced abelian p-groups having the same Ulm factors might differ.

In what follows, all groups will be p-primary abelian groups. If α is an ordinal, by $\beta \in \alpha$ we mean $0 \leqq \beta < \alpha$, ω is the first infinite ordinal, $\omega^* = \omega - \{0\}$, and c is the cardinality of the continuum. If G is a group and $a \in G$, then $\tau(G)$ will be the Ulm type of G, $r(G)$ the rank of G, and $h(a)$ the generalized height of a in G (see [4]). Also $\oplus$ and $\sum$ will be used for direct sum and $\langle \cdots \rangle$ for group generated by $\cdots$. In general, the notation and terminology will be similar to that of [4] or [10].

1. **Preliminaries.** For notational purposes we will say that a subgroup H of G is β-high in G (β an ordinal) if H is a $p^\beta G$-high subgroup of G (see [7]). Thus an ω-high subgroup of G is just a high subgroup of G.

The following lemma will be very useful in the sequel.

LEMMA 1.1. *Let G be a reduced p-group and let H be a $(\alpha+n)$-high subgroup of G with α a limit ordinal and n a nonnegative integer. Then $G[p^s] = H[p^s] \oplus K[p^s]$ for $s \leqq n+1$ and any complementary summand K of a maximal p^n-bounded summand of $p^\alpha G$.*

Proof. Clearly $G[p] = H[p] \oplus K[p]$. Suppose $G[p^s] = H[p^s] \oplus K[p^s]$ for $s < n+1$. Let $x \in G[p^{s+1}] - G[p^s]$. Then $px = h + y$ such that $h \in H[p^s]$ and $y \in K[p^s]$. Since K is a complementary summand of a maximal p^n-bounded summand of $p^\alpha G$ we may choose $z \in K$ such that $pz = y$. Since H is pure we may choose $k \in H$ such that $pk = h$. Thus $x - (k+z) \in G[p]$ and the result follows.

The next theorem arose while considering the following question: Let G and H be groups with high subgroups K and L, respectively. Let $\alpha\colon G \to G/p^\omega G$ and $\beta\colon H \to H/p^\omega H$ be natural. Suppose that $p^\omega G \cong p^\omega H$ and that γ is an isomorphism from K onto L such that the isomorphism induced by γ from $\alpha(K)$ onto $\beta(L)$ can be extended to an isomorphism from $G/p^\omega G$ onto $H/p^\omega H$. Are G and H isomorphic? If $p^\omega G$ is a direct sum of cyclic groups each of order p^n then the answer is in the affirmative as indicated by the following theorem. In general the answer is negative as is shown by Example 1 in §4.

THEOREM 1.2. *Let G and H be reduced p-groups, δ a limit ordinal, and n a positive integer such that*

(i) *there exist $(\delta+n-1)$-high subgroups G_{n-1} and H_{n-1} of G and H, respectively, that are isomorphic under an isomorphism α and*

(ii) *there exist $(\delta+n)$-high subgroups G_n and H_n of G and H, respectively, containing G_{n-1} and H_{n-1}, respectively, and an isomorphism σ from $G_n/p^\delta G_n$ onto $H_n/p^\delta H_n$ such that $\sigma(x+p^\delta G_n) = \alpha(x) + p^\delta H_n$ for all $x \in G_{n-1}$.*

Then there exists an isomorphism β from G_n onto H_n such that $\sigma(x+p^\delta G_n) = \beta(x) + p^\delta H_n$ for all $x \in G_n$ and $\beta | G_{n-1} = \alpha$.

Proof. Note that $p^\delta G_k=\sum_{i=1}^k B_i$ with $B_i=\sum_{\mu\in\Gamma_i}\langle u_i^\mu\rangle$ such that $O(u_i^\mu)=p^i$ and $p^\delta H_k=\sum_{i=1}^k C_i$ with $C_i=\sum_{\mu\in\Gamma_i}\langle v_i^\mu\rangle$ such that $O(v_i^\mu)=p^i$, for $k=n-1, n$. (The rank of B_n is equal to the rank of C_n since the rank of G_n/G_{n-1} is equal to the rank of H_n/H_{n-1}.)

For each $\mu\in\Gamma_n$ there exists a sequence $\{x_i^\mu\}_{i\in\omega}$ of elements of G_n satisfying the following conditions:

1. $x_0^\mu = u_n^\mu$,
2. $p^{ni}x_i^\mu = x_0^\mu$ for all $i\in\omega$,
3. $x_i^\mu-p^n x_{i+1}^\mu\in G_{n-1}$ for all $i\in\omega$,
4. $G_n/G_{n-1} = \langle\{x_i^\mu+G_{n-1} \mid i\in\omega, \mu\in\Gamma_n\}\rangle$.

For each $\mu\in\Gamma_n$ we construct such a sequence inductively. Let $x_0^\mu=u_n^\mu$ and choose $x_1^\mu\in G_n$ such that $p^n x_1^\mu=x_0^\mu$. Assume that $x_0^\mu,\dots,x_i^\mu$ have been obtained. Since G_n/G_{n-1} is divisible there exists $y\in G_n$ such that $x_i^\mu+G_{n-1}=p^n y+G_{n-1}$. Thus $x_0^\mu+G_{n-1}=p^{ni}x_i^\mu+G_{n-1}=p^{n(i+1)}y+G_{n-1}$. Since G_{n-1} is pure in G_n and $h(x_0^\mu)\geqq\omega$, there exists $g\in G_{n-1}$ such that $x_0^\mu=p^{n(i+1)}(y+g)$. Let $x_{i+1}^\mu=y+g$. Then $p^{n(i+1)}x_{i+1}^\mu$ $=x_0^\mu$ and $x_i^\mu-p^n x_{i+1}^\mu=(x_i^\mu-p^n y)-p^n g\in G_{n-1}$. Thus conditions 1–4 are satisfied.

Next, for each $\mu\in\Gamma_n$, there exists a sequence $\{y_i^\mu\}_{i\in\omega}$ of elements of H_n satisfying the following conditions:

1′. $p^\delta H_n = \sum_{i=1}^{n-1} C_i \oplus \sum_{\mu\in\Gamma_n}\langle y_0^\mu\rangle$ such that $O(y_0^\mu)=p^n$,

2′. $p^{ni}y_i^\mu = y_0^\mu$ for all $i\in\omega$,

3′. $y_i^\mu\in\sigma(x_i^\mu+p^\delta G_n)$,

4′. $y_i^\mu-p^n y_{i+1}^\mu = \alpha(x_i^\mu-p^n x_{i+1}^\mu)\in H_{n-1}$,

5′. $H_n/H_{n-1} = \langle\{y_i^\mu+H_{n-1} \mid i\in\omega, \mu\in\Gamma_n\}\rangle$.

Let $\mu\in\Gamma_n$. Choose $w_i^\mu\in\sigma(x_i^\mu+p^\delta G_n)$ for each $i\in\omega^*$. Note that

$$O(w_i^\mu+p^\delta H_n) = O(\sigma^{-1}(w_i^\mu+p^\delta H_n)) = O(x_i^\mu+p^\delta G_n) = p^{ni}.$$

Thus $p^{ni}\leqq O(w_i^\mu)\leqq p^{n(i+1)}$. Then

$$p^{ni}w_i^\mu = k+p^s\sum_{\mu\in\Gamma_n} a_\mu v_n^\mu$$

such that $k\in p^\delta H_{n-1}$, $0\leqq s\leqq n$, $a_\mu=0$ for almost all $\mu\in\Gamma_n$, and there exists $\eta\in\Gamma_n$ such that $0<a_\eta<p$. Suppose $s>0$. Let $k'\in H_{n-1}$ such that $pk'=k$. Then

$$p^{ni-1}w_i^\mu-p^{s-1}\sum_{\mu\in\Gamma_n} a_\mu v_n^\mu+k' = h+m,$$

$h\in(H_{n-1})[p]$ and $m\in C_n[p]$. Thus

$$p^{ni-1}x_i^\mu+p^\delta G_n = \sigma^{-1}(p^{ni-1}w_i^\mu+p^\delta H_n) = \alpha^{-1}(h-k')+p^\delta G_n.$$

Therefore $p^{ni-1}x_i^\mu=\alpha^{-1}(h-k')+g$ for some $g\in p^\delta G_n$. This implies that $\alpha^{-1}(-k)+pg$ $=p^{ni}x_i^\mu=x_0^\mu$ which is a contradiction. Thus $O(w_i^\mu)=p^{n(i+1)}$.

Let $y_0^\mu=p^n w_1^\mu$ for each $\mu\in\Gamma_n$. Then $\sum_{\mu\in\Gamma_n}\langle y_0^\mu\rangle$ is direct and

$$p^\delta H_n = \sum_{i=1}^{n-1} C_i \oplus \sum_{\mu\in\Gamma_n}\langle y_0^\mu\rangle.$$

To show that the sum is direct, first note that any $y \in \langle\{y_0^\mu \mid \mu \in \Gamma_n\}\rangle$ may be expressed as $y = p^s \sum_{\mu\in\Gamma_n} a_\mu y_0^\mu$, where $0 \leqq s \leqq n$, $0 \leqq a_\mu < p^n$ for all $\mu \in \Gamma_n$, $a_\mu = 0$ for almost all $\mu \in \Gamma_n$, and $0 < a_\eta < p$ for some $\eta \in \Gamma_n$. Suppose that $0 = p^s \sum_{\mu\in\Gamma_n} a_\mu y_0^\mu$ with $s < n$. Then from Lemma 1.1 we have $p^{n-1} \sum_{\mu\in\Gamma_n} a_\mu w_1^\mu = h + k$ such that $h \in (H_{n-1})[p^{s+1}]$ and $k \in C_n[p^{s+1}]$. Thus

$$p^{n-1} \sum_{\mu\in\Gamma_n} a_\mu x_1^\mu + p^\delta G_n = \sigma^{-1}\Big(p^{n-1} \sum_{\mu\in\Gamma_n} a_\mu w_1^\mu + p^\delta H_n\Big) = \alpha^{-1}(h) + p^\delta G_n.$$

Hence $p^{n+s} \sum_{\mu\in\Gamma_n} a_\mu x_1^\mu = p^{s+1}\alpha^{-1}(h) + p^{s+1}g = p^{s+1}g$ for some $g \in p^\delta G_n$. But the height of $p^{n+s} \sum_{\mu\in\Gamma_n} a_\mu x_1^\mu$ in $p^\delta G_n$ is s (since $0 < a_\eta < p$) which is a contradiction. Thus the sum is direct. That $p^\delta H_n = \sum_{i=1}^{n-1} C_i \oplus \sum_{\mu\in\Gamma_n} \langle y_0^\mu\rangle$ follows from the fact that σ is an isomorphism onto $H_n/p^\delta H_{n-1}$.

Assume that $y_0^\mu, \ldots, y_{i-1}^\mu$ have been defined such that conditions 2′, 3′, and 4′ are satisfied, $y_j^\mu - w_j^\mu \in p^\delta G_n$ for $0 < j < i$, and $\alpha(x_{i-1}^\mu - p^n x_i^\mu) = y_{i-1}^\mu - p^n w_i^\mu$. Choose $y_i^\mu = w_i^\mu - a$ where $a = w_i^\mu - p^n w_{i+1}^\mu - \alpha(x_i^\mu - p^n x_{i+1}^\mu) \in p^\delta G_n$. It is easily seen that the sequences $\{y_i^\mu\}_{i\in\omega}$, $\mu \in \Gamma_n$, satisfy conditions 1′–5′.

With the sequence just constructed we can extend α to an isomorphism from G_n onto H_n. First we note that if $x \in G_n$, then x can be expressed uniquely as $x = c + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} a_i^\mu x_i^\mu$ where $0 \leqq a_i^\mu < p^n$, $c \in G_{n-1}$, and $a_i^\mu = 0$ for almost all $i \in \omega$ and $\mu \in \Gamma_n$. This follows from Theorem 29.7 in [4]. Thus the desired extension is obtained by defining

$$\begin{aligned}\beta\colon G_n \to H_n\colon \beta(x) &= \beta\Big(c + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} a_i^\mu x_i^\mu\Big)\\ &= \alpha(c) + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} a_i^\mu y_i^\mu.\end{aligned}$$

It is clear that β is a well-defined, one-to-one, onto map. Let $x, y \in G_n$. Then $x = c + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} a_i^\mu x_i^\mu$ and $y = d + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} b_i^\mu x_i^\mu$ where $c, d \in G_{n-1}$ and a_i^μ, b_i^μ are restricted as above. In order to consider the sum $x+y$ we develop the following. Let $\mu \in \Gamma_n$. If $a_i^\mu, b_i^\mu = 0$ for all $i \in \omega$ define $c_i^\mu = 0$ for all $i \in \omega$. If not, let j be the largest integer such that a_j^μ or $b_j^\mu \neq 0$. There exists unique integers m_j and r_j such that $a_j + b_j = m_j p^n + r_j$ with $0 \leqq r_j < p^n$ $(m_j = 0, 1)$. Assuming m_{j-i+1}, r_{j-i+1} have been obtained for $i \leqq j$, let r_{j-i} and m_{j-i} be the unique integers such that $a_{j-i} + b_{j-i} + m_{j-i+1} = m_{j-i}p^n + r_{j-i}$ with $0 \leqq r_{j-i} < p^n$ $(m_{j-i} = 0, 1)$. Let $c_i^\mu = r_i$ for $i = 0, \ldots, j$ and $c_i^\mu = 0$ for $i > j$. If $m_{j-i} \neq 0$, $0 \leqq i < j$, let $d_i^\mu = p^n x_{j-i}^\mu - x_{j-i-1}^\mu$. Otherwise let $d_i^\mu = 0$. Then $x + y = c + d + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} (c_i^\mu x_i^\mu + d_i^\mu)$ where $d_i^\mu \in G_{n-1}$. Thus

$$\beta(x+y) = \alpha(c+d) + \sum_{\mu\in\Gamma_n} \sum_{i\in\omega} (c_i^\mu y_i^\mu + \alpha(d_i^\mu)) = \beta(x) + \beta(y)$$

where the second equality follows from condition 4′. Therefore β is the desired isomorphism.

The following lemma will be useful in determining how the Ulm factors are put together to form the group.

LEMMA 1.3. *Let G be a reduced p-group, δ a limit ordinal, and $\{G_i\}_{i\in\omega}$ a chain $(G_i\subset G_{i+1})$ of $(\delta+i)$-high subgroups of G. Let $G_\delta=\bigcup_{i\in\omega} G_i$. Then $G=\langle G_\delta, p^\delta G\rangle$. $p^\delta G=\bigcup_{i\in\omega} p^\delta G_i$ if and only if $G=\bigcup_{i\in\omega} G_i$. Also $p^\delta G_\delta$ is a basic subgroup of $p^\delta G$.*

Proof. Since G_n is isotype (see [8]), $p^\delta G_n=\sum_{i=1}^n B_i$, such that B_i is a direct sum of cyclic groups of order p^i; $B=\sum_{i\in\omega} B_i$ is a basic subgroup of $p^\delta G$; and $p^\delta G=\sum_{i=1}^n B_i \oplus K_n$ where $K_n=\langle \sum_{i=n+1}^\infty B_i, p^{\delta+n} G\rangle$. Let $x\in G$ with order p^{k+1}. By Lemma 1.1, $x=a+b$ with $a\in G_k[p^{k+1}]$ and $b\in K_k[p^{k+1}]$. Thus $x\in\langle G_\delta, p^\delta G\rangle$. The lemma follows.

COROLLARY 1.4. *$G=\langle G_\delta, p^{\delta+n}G\rangle$, n an integer.*

COROLLARY 1.5. *$G/p^\delta G\cong G_\delta/p^\delta G_\delta$.*

Proof. $G/p^\delta G=(G_\delta+p^\delta G)/p^\delta G\cong G_\delta/(G_\delta\cap p^\delta G)=G_\delta/p^\delta G_\delta$. ($G_\delta\cap p^\delta G=p^\delta G_\delta$ since G_δ is isotype.)

COROLLARY 1.6. *Using the notation of Lemma 1.3 suppose there exists isomorphisms α from G_δ onto H_δ and β from $p^\delta G$ onto $p^\delta H$ such that $\alpha|p^\delta G_\delta=\beta|p^\delta G_\delta$. Then there exists an (unique) isomorphism φ from G onto H such that $\varphi|G_\delta=\alpha$ and $\varphi|p^\delta G=\beta$.*

Proof. If $g\in G$ then $g=a+b$ with $a\in G_\delta$ and $b\in p^\delta G$. Thus define

$$\varphi\colon G\to H\colon \varphi(g) = \alpha(a)+\beta(b).$$

It is easily seen that φ is an isomorphism.

THEOREM 1.7. *Let G and H be reduced p-groups and δ a limit ordinal such that $p^\delta G$ and $p^\delta H$ are torsion complete. A necessary and sufficient condition for G and H to be isomorphic is that there exist sequences $\{G_i\}_{i\in\omega}$ and $\{H_i\}_{i\in\omega}$ of subgroups of G and H, respectively, satisfying the following conditions:*

1. *For each $i\in\omega$, G_i and H_i are $(\delta+i)$-high subgroups of G and H, respectively.*
2. *For each $i\in\omega$, $G_i\subset G_{i+1}$ and $H_i\subset H_{i+1}$.*
3. *There exists a sequence $\{\varphi_i\}_{i\in\omega}$ such that φ_i is an isomorphism from $G_i/p^\delta G_i$ onto $H_i/p^\delta H_i$ such that if $i>0$, $\varphi_i|G_{i-1}/p^\delta G_{i-1}=\varphi_{i-1}$ where $G_{i-1}/p^\delta G_{i-1}$ is embedded in $G_i/p^\delta G_i$ in the natural manner.*
4. *$p^\delta G$ and $p^\delta H$ are isomorphic.*

Proof. The necessity is clear. For the sufficiency we first construct a sequence $\{\hat\varphi_i\}_{i\in\omega}$ such that $\hat\varphi_i$ is an isomorphism from G_i onto H_i, $\hat\varphi_i|G_{i-1}=\hat\varphi_{i-1}$, $i>0$, and $\varphi_i(x+p^\delta G_i)=\hat\varphi_i(x)+p^\delta H_i$ for all $x\in G_i$. We proceed by induction. Let $\hat\varphi_0=\varphi_0$. Assuming that $\hat\varphi_{i-1}$ has been constructed, note that $\varphi_i(x+p^\delta G_i)=\hat\varphi_{i-1}(x)+p^\delta H_i$ for all $x\in G_{i-1}$. Thus by Theorem 1.2 there exists an isomorphism $\hat\varphi_i$ from G_i onto H_i such that $\varphi_i(x+p^\delta G_i)=\hat\varphi_i(x)+p^\delta H_i$ and $\hat\varphi_i|G_{i-1}=\hat\varphi_{i-1}$. Let $G_\delta=\bigcup_{i\in\omega} G_i$ and $H_\delta=\bigcup_{i\in\omega} H_i$. Let φ be an isomorphism from G_δ onto H_δ such that $\varphi|G_i=\hat\varphi_i$. Let β be an isomorphism from $p^\delta G$ onto $p^\delta H$ such that $\beta|p^\delta G_\delta=\varphi|p^\delta G_\delta$. ($\beta$ exists since

$p^\delta G$ is torsion complete and $p^\delta G_\delta$ is a basic subgroup of $p^\delta G$ (see [11]).) By Corollary 1.6, there exists an isomorphism γ from G onto H such that $\gamma|G_\delta=\varphi$ and $\gamma|p^\delta G=\beta$. Thus G and H are isomorphic.

If we define the chain of subgroups $\{(G_i+p^\delta G)/p^\delta G\}_{i\in\omega}$ to be a "δ-high chain" in $G/p^\delta G$ then, in the case that $p^\delta G$ is torsion complete, G is characterized by its class of "δ-high chains" and $p^\delta G$.

COROLLARY 1.8. *Let G and H be p-groups such that $G/p^\omega G\cong H/p^\omega H$, $G/p^\omega G$ is torsion complete, $p^\omega G\cong p^\omega H\cong \sum Z(p^n)$ (n fixed), and G and H have isomorphic high subgroups. Then G and H are isomorphic.*

Under the conditions of Corollary 1.8 we have that G is characterized by $G/p^\omega G$, $p^\omega G$, and its class of high subgroups.

LEMMA 1.9. *Let G be a reduced p-group, H a subgroup of G with basic subgroup B, and α an ordinal. If $B\subset p^\alpha G$ then $H\subset p^\alpha G$.*

Proof. Assume that $H\subset p^\beta G$ for all $\beta<\sigma\leqq\alpha$. Let $x\in H$. If σ is not a limit ordinal let δ be such that $\sigma=\delta+1$. Then $x\in p^\delta G$, and since B is basic in H there exists $y\in H\subset p^\delta G$ such that $py-x=b\in B$. Since $B\subset p^\alpha G\subset p^\sigma G$ there exists $z\in p^\delta G$ such that $pz=b$. Thus $p(y-z)=x$, and $x\in p^\sigma G$. If σ is a limit ordinal then $x\in p^\beta G$ for all $\beta<\sigma$ and hence $x\in p^\sigma G$. Thus $H\subset p^\sigma G$. Therefore $H\subset p^\alpha G$.

LEMMA 1.10. *Let G be a reduced p-group and let $\{\alpha_j\}_{j\in\omega}$ be an increasing sequence of ordinals such that $\alpha_0=0$. For each $j\in\omega$, let β_j be the ordinal such that $\omega\alpha_j+\beta_j=\omega\alpha_{j+1}$. For each $j\in\omega$ let $\{K_i^j\}_{i\in\omega}$ be a chain of subgroups of G such that K_i^j is (β_j+1)-high in $p^{\omega\alpha_j}G$, and $K_0^{j+1}\supset p^{\beta_j}K_\omega^j$ where $K_\omega^j=\bigcup_{i\in\omega}K_i^j$. Then $S_i^n=\langle\{K_\omega^j\mid j<n\}, K_i^n\rangle$ is $(\omega\alpha_{n+1}+i)$-high in G.*

Proof. Well order $\{(n,i)\mid n,i\in\omega\}$ by $(n,i)<(m,k)$ if $n<m$ and $(n,i)<(m,k)$ if $n=m$ and $i<k$. If $n=0$ then clearly S_i^0 is $(\omega+i)$-high in G for all $i\in\omega$. Suppose the result is true for all $(n,i)<(m,k)$, $m>0$. Note that $S_k^m\cap p^{\omega\alpha_{m+1}+k}G=0$, and let S be an $(\omega\alpha_{m+1}+k)$-high subgroup of G containing S_k^m. Let $x\in S$ and suppose $O(x)=p^{r+1}$ for some $r\in\omega$. First note that by Lemma 1.9 and the induction hypothesis we have $p^{\omega\alpha_m}S_k^m=K_k^m$. Next note that $S_k^m[p^{r+1}]=S_r^{m-1}[p^{r+1}]\oplus M[p^{r+1}]$ where M is a complementary summand of a maximal p^r-bounded summand of K_k^m. This follows from the fact that $p^{\omega\alpha_m}S_k^m=K_k^m$ and Lemma 1.1. Let B be a basic subgroup of M and $L=\langle B, p^{\omega\alpha_m+r}G\rangle$. Then $M\subset L$ and $G[p^{r+1}]=S_r^{m-1}[p^{r+1}]\oplus L[p^{r+1}]$. (Again the induction hypothesis and Lemma 1.1 are used.) Now $x=a+b$ such that $a\in S_r^{m-1}[p^{r+1}]$ and $b\in L[p^{r+1}]$. Thus $b\in S\cap p^{\omega\alpha_m}G$ and it follows that $\langle b, K_k^m\rangle\cap p^{\omega\alpha_{m+1}+k}G=0$. Since K_k^m is (β_m+k)-high in $p^{\omega\alpha_m}G$, $b\in K_k^m$. Therefore $S=S_k^m$.

COROLLARY 1.11. *S_i^n is isotype in G.*

COROLLARY 1.12. *Following the notation of Lemma* 1.10, *the map*

$$\sigma: S_k^m/p^{\omega\alpha_m}S_k^m \to G/p^{\omega\alpha_m}G: x+p^{\omega\alpha_m}S_k^m \to x+p^{\omega\alpha_m}G$$

is an isomorphism of $S_k^m/p^{\omega\alpha_m}$ *onto* $G/p^{\omega\alpha_m}G$.

Proof. Let $x \in G$ such that $O(x)=p^{r+1}$. Then $G[p^{r+1}]=S_r^{m-1}[p^{r+1}] \oplus L[p^{r+1}]$ and $x=a+b$ with $a \in S_r^{m-1}[p^{r+1}]$ and $b \in L[p^{r+1}]$. Since $L\subset p^{\omega\alpha_m}G$, $\sigma(a+p^{\omega\alpha_m}S_k^m)$ $=x+p^{\omega\alpha_m}G$. Hence σ is onto. The result follows easily.

COROLLARY 1.13. *Following the notation of Lemma* 1.10,

$$K_\omega^j/p^{\beta_j}K_\omega^j \cong p^{\omega\alpha_j}G/p^{\omega\alpha_{j+1}}G.$$

REMARK 1.14 (TOPOLOGICAL CONSIDERATIONS). Let G be a reduced p-group, β an ordinal $\geqq 1$. Let $\mathscr{T}^\beta$ be the topology induced on G by taking the chain of subgroups $\{p^\alpha G\}_{\alpha\in\omega\beta}$ as a fundamental system of neighborhoods of 0. Let $\mathscr{T}_n$ be the relative topology induced on $G[p^n]$ by $\mathscr{T}^\beta$. Let $\mathscr{T}_\beta$ be the inductive limit topology on G obtained by taking the inductive limit of the topological groups $(G[p^n], \mathscr{T}_n)$ in the category of topological abelian groups (see [2]). We shall refer to $\mathscr{T}_\beta$ as the β-topology on G. Note that $\mathscr{T}_\beta$ is Hausdorff if the Ulm type of G is $\leqq\beta$, and, in this case, the completion of G with respect to the β-topology is the torsion subgroup of the completion of G with respect to $\mathscr{T}^\beta$.

To obtain a convenient fundamental system of neighborhoods of zero for the β-topology we need a special class of I. Kaplansky's U-sequences (see [10]). Let G be a reduced p-group of Ulm type $\geqq\lambda$ (λ a limit ordinal). A sequence $\tilde{\alpha}=\{\alpha_i\}_{i\in\omega}$ with $\alpha_0, \alpha_1, \alpha_2, \ldots$ ordinals less than $\omega\lambda$ is called a $(\lambda-U)$-sequence for G if

(i) $\alpha_0<\alpha_1<\alpha_2<\cdots$, and

(ii) $\alpha_k+1<\alpha_{k+1}$ implies $f_G(d_k)\neq 0$ where $f_G(\alpha_k)$ is the α_kth Ulm invariant of G.
The Ulm sequence of an element $x \in G$ is defined to be $U_G(x)=\{h_G(p^i x)\}_{i\in\omega}$ (the height of $p^i x$ in G).

Partially order these sequences componentwise. If $\tilde{\alpha}$ is a $(\lambda-U)$-sequence for G, then $G\{\tilde{\alpha}\}=\{x \in G \mid U_G(x)\geqq\tilde{\alpha}\}$ is an unbounded fully invariant subgroup of G. Let K be the set of all $(\lambda-U)$-sequences for G. Let $\mathscr{F}=\{G\{\tilde{\alpha}\} \mid \tilde{\alpha}\in K\}$. Then $\mathscr{F}$ is a fundamental system of neighborhoods of zero for $\mathscr{T}_\beta$.

REMARK 1.15 (SUBDIRECT SUM). Let G be an unbounded reduced p-group of Ulm type $\beta\geqq 1$. If G has Ulm type 1 we endow G with the large topology (see [3]). If G has Ulm type β we endow G with the β-topology. Let H be a dense isotype subgroup of G; let B be a direct sum of cyclic p-groups such that B has the same rank as G/H; let D be a divisible hull of B; let $\sigma: G \to G/H$ and $\gamma: D \to D/B$ be the natural homomorphisms; and let f be an isomorphism from G/H onto D/B. Define $K=\{(g, d) \mid g \in G,\ d \in D$, and $f(\sigma(g))=\gamma(d)\}$, a subgroup of $G \oplus D$. It can be shown that $H \oplus \{0\}$ is an $\omega\beta$-high subgroup of K and $\{0\} \oplus B=p^{\omega\beta}K$. We shall identify H with $H \oplus \{0\}$ and B with $\{0\} \oplus B$. We shall refer to K as the *subdirect sum of* G *and* D *with* $\omega\beta$*-high subgroup* H *and* $p^{\omega\beta}K=B$.

Let $B=\sum_{i\in\omega} B_i$ with $B_i=\sum_{\alpha\in\Gamma_i}\langle x_{\alpha,i}\rangle$ and $O(x_{\alpha,i})=p^i$ for all $\alpha\in\Gamma_i$, and let $\{H_i\}_{i\in\omega}$ be a chain of dense isotype subgroups of G such that $\bigcup_{i\in\omega} H_i=G$ and H_{i+1}/H_i has the same rank as B_i. Let D_i be a divisible hull of B_i for each $i\in\omega$, $D=\sum_{i\in\omega} D_i$, and choose a $y_{\alpha,i}\in D_i$, for each $\alpha\in\Gamma_i$ and $i\in\omega^*$, such that $py_{\alpha,i}=x_{\alpha,i}$. Let $L_i=\sum_{\alpha\in\Gamma_i}\langle z_{\alpha,i}\rangle$ be a complementary summand of $H_i[p]$ in $H_{i+1}[p]$ for each $i\in\omega$. Let $\sigma: G\to G/H_0$ and $\mathscr{T}_i: D_i\to D_i/B_i$ for each $i\in\omega$ be natural homomorphisms, and let f be an isomorphism from G/H_0 onto $\sum_{i\in\omega} D_i/B_i$ such that $f|H_j/H_0=\sum_{i=1}^{j} D_i/B_i$ and $f(z_{\alpha,i}+H_0)=y_{\alpha,i}+\sum B_i$. Define $K=\{(g,d)\mid g\in G, d\in D$, and $f(\sigma(g))=\sum \mathscr{T}_i(d_i)$ where $d=\sum d_i$, $d\in D_i\}$, a subgroup of $G\oplus D$. It is easily shown that $H_i'=\{(h,d)\mid h\in H_i, d\in\sum_{j=0}^{i} D_j$, and $f(\sigma(h))=\sum \gamma_j(d_j)$ where $d=\sum d_j$, $d_j\in D_j\}$ is $(\omega\beta+i)$-high in K and $p^{\omega\beta}K=B$. In this case we shall refer to K as *the subdirect sum of G and D with "β-high chain"* $\{H_i\}_{i\in\omega}$ *and* $p^{\omega\beta}K=B$.

REMARK 1.16. If $\alpha>\omega$ is a countable limit ordinal, then α is of one and only one of the following types:

Type 1. There exists a limit ordinal $\beta<\alpha$ such that $\beta\omega>\alpha$.

Type 2. There exists a sequence $\{\alpha_i\}_{i\in\omega}$ of countable limit ordinals such that $\alpha_0=0$, $\alpha_{i+2}-\alpha_{i+1}\geqq\alpha_{i+1}-\alpha_i$ for all $i\in\omega$, and $\lim_{i\in\omega}\alpha_i=\alpha$.

Note that if α is of type 1 we may choose β of type 2, and if α is of type 2, we may choose the α_i's of type 2. Hence we will call a sequence $\{\alpha_i\}_{i\in\omega}$ of countable limit ordinals an *admissible sequence* if $\alpha_0=0$, $\alpha_{i+2}-\alpha_{i+1}\geqq\alpha_{i+1}-\alpha_i$ for all $i\in\omega$, and α_i is of type 2 for all $i\in\omega$.

2. **Reduced p-primary abelian groups of Ulm type $\leqq\omega$.** We will first consider groups of finite type and determine their structure in terms of the Ulm factors, sequences of subgroups in each Ulm factor, and sequences of sets of elements of each Ulm factor that determine how the Ulm factors were "connected." We will then define an ω-summable group whose structure is determined in a similar manner, induce a topology on these groups, and show that every group of Ulm type ω is an isotype subgroup of the completion of an ω-summable group.

DEFINITION 2.1. Let G be a reduced p-group and α an ordinal number such that $\alpha<\tau(G)$. Let $\{K_i\}_{i\in\omega}$ be a chain of subgroups of $p^{\omega\alpha}G$ such that K_i is $(\omega+i)$-high in $p^{\omega\alpha}G$. Let $\sigma: p^{\omega\alpha}G\to G_\alpha=p^{\omega\alpha}G/p^{\omega(\alpha+1)}G$ be the natural homomorphism. Then $\{K_i\}_{i\in\omega}$ will be called a high chain of $p^{\omega\alpha}G$ and $\{\sigma(K_i)\}_{i\in\omega}$ will be called a high chain of G_α.

DEFINITION 2.2. Let G and H be reduced p-groups and let G_α, $\alpha\in\tau(G)$, and H_α, $\alpha\in\tau(H)$, be the αth Ulm factors of G and H, respectively. Let $\varphi_i: G_{\alpha+i}\to H_{\alpha+i}$, $i\in\omega$, be (onto) isomorphisms. Then $\{\varphi_i\}_{i\in\omega}$ is said to be compatible if for every $j\in\omega$ there exist

(i) high chains $\{K_i^j\}_{i\in\omega}$ and $\{L_i^j\}_{i\in\omega}$ of $p^{\omega(\alpha+j)}G$ and $p^{\omega(\alpha+j)}H$, respectively,

(ii) decompositions $p^\omega K_i^j=\sum_{k=1}^{i}\sum_{\mu\in\Gamma_{(k,j)}}\langle x_k^{(\mu,j)}\rangle$ and $p^\omega L_i^j=\sum_{k=1}^{i}\sum_{\mu\in\Gamma_{(k,j)}}\langle y_k^{(\mu,j)}\rangle$ for each $i\in\omega^*$, and

(iii) $\{\{u_i^{(\mu,j)}\mid\mu\in\Gamma_{(i,j)}\}\}_{i\in\omega^*}$, $u_i^{(\mu,j)}\in K_i^j$, and $\{\{v_i^{(\mu,j)}\mid\mu\in\Gamma_{(i,j)}\}\}_{i\in\omega^*}$, $v_i^{(\mu,j)}\in L_i^j$ satisfying the following conditions:

(a) For all $j \in \omega$, if $K_\omega^j = \bigcup_{i\in\omega} K_i^j$ and $L_\omega^j = \bigcup_{i\in\omega} L_i^j$ then $p^\omega K_\omega^j \subset K_0^{j+1}$ and $p^\omega L_\omega^j \subset L_0^{j+1}$.

(b) For all $\mu \in \Gamma_{(i,j)}$, $i \in \omega^*$, and $j \in \omega$, $p^i u_i^{(\mu,j)} = x_i^{(\mu,j)}$ and $p^i v_i^{(\mu,j)} = y_i^{(\mu,j)}$.

(c) For all $\mu \in \Gamma_{(i,j)}$, $i \in \omega$, and $j \in \omega$, if $\sigma_j: p^{\omega(\alpha+j)}G \to G_{\alpha+j}$ and $\rho_j: p^{\omega(\alpha+j)}H \to H_{\alpha+j}$ are the natural homomorphisms, then $\varphi_j\sigma_j(u_i^{(\mu,j)}) = \rho_j(v_i^{(\mu,j)})$, $\varphi_{j+1}\sigma_{j+1}(x_i^{(\mu,j)}) = \rho_{j+1}(y_i^{(\mu,j)})$, and $\varphi_j\sigma_j(K_i^j) = \rho_j(L_i^j)$.

We state the following structure theorem in spite of its inelegance because of its structural value.

THEOREM 2.3. *Let G and H be reduced p-groups with at most a finite number of Ulm factors $G_0, G_1, \ldots, G_m$ and $H_0, H_1, \ldots, H_m$, respectively. A necessary and sufficient condition for G and H to be isomorphic is that there exist isomorphisms $\varphi_j: G_j \to H_j$, $j = 0, \ldots, m$, such that $\{\varphi_j\}_{j=0}^m$ is compatible.*

Proof. Since the necessity is clear we will prove only the sufficiency. We will use the notation of Definition 2.2 with $\alpha = 0$.

Let $\sigma: G \to G/p^{\omega 2}G$ and $\rho: H \to H/p^{\omega 2}H$ be the natural homomorphisms. First we construct an isomorphism α_0 from $G/p^{\omega 2}G$ onto $H/p^{\omega 2}H$ such that $\alpha_0\sigma(u_i^{(\mu,0)}) = \rho(v_i^{(\mu,0)})$ for all $i \in \omega^*$, $\mu \in \Gamma_{(i,0)}$. Construct, inductively on n, an isomorphism α from $\sigma(K_\omega^0)$ onto $\rho(L_\omega^0)$ such that in the nth step we use the construction in the proof of Theorem 1.2 and choose $x_1^\mu = \sigma(u_n^{(\mu,0)})$ and $w_1^\mu = \rho(v_n^{(\mu,0)})$ for all $\mu \in \Gamma_{(n,0)}$. Thus $\alpha|\sigma(p^\omega K_\omega^0) = \varphi_1|\sigma(p^\omega K_\omega^0)$. By Corollary 1.6 there exists an isomorphism α_0 from $G/p^{\omega 2}G$ onto $H/p^{\omega 2}H$ such that $\alpha_0|\sigma(K_\omega^0) = \alpha$ and $\alpha_0|G_1 = \varphi_1$.

Next let $\sigma: G \to G/p^{\omega(k+2)}G$ and $\rho: H \to H/p^{\omega(k+2)}H$ be natural. Let

$$S_i^k = \langle \{K_\omega^j \mid j < k\}, K_i^k \rangle \quad \text{and} \quad T_i^k = \langle \{L_\omega^j \mid j < k\}, L_i^k \rangle$$

for all $i \in \omega$ and $S_\omega^k = \bigcup_{i\in\omega} S_i^k$ and $T_\omega^k = \bigcup_{i\in\omega} T_i^k$ (as considered in Lemma 1.10). Suppose an isomorphism α_k has been defined from $G/p^{\omega(k+1)}G$ onto $H/p^{\omega(k+1)}H$ such that $\alpha_k|G_k = \varphi_k$. Construct, inductively on n, an isomorphism α from $\sigma(S_\omega^k)$ onto $\rho(T_\omega^k)$ such that in the nth step we use the construction in the proof of Theorem 1.2 and choose $x_1^\mu = \sigma(u_n^{(\mu,k+1)})$ and $w_1^\mu = \rho(v_n^{(\mu,k+1)})$ for all $\mu \in \Gamma_{(n,k+1)}$. Thus $\alpha|\sigma(p^{\omega(k+1)}S_\omega^k) = \varphi_{k+1}|\sigma(p^{\omega(k+1)}S_\omega^k)$. By Corollary 1.6, there is an isomorphism α_{k+1} from $G/p^{\omega(k+2)}$ onto $H/p^{\omega(k+2)}H$ such that $\alpha_{k+1}|\sigma(S_\omega^k) = \alpha$ and $\alpha_{k+1}|G_{k+1} = \varphi_{k+1}$. Thus we obtain an isomorphism α_{m-1} from G onto H.

COROLLARY 2.4. *Any reduced p-group with a finite number of Ulm factors can be constructed from the Ulm factors using the methods of Theorem* 38.1 *in* [4].

DEFINITION 2.5. Let G be a reduced p-group. Then G will be said to be ω-summable if there exists a chain of subgroups $\{G_i\}_{i\in\omega^*}$, with G_i ωi-high in G, such that $G = \bigcup_{i\in\omega^*} G_i$.

NOTATION 2.6. Let α and β be limit ordinals, and $i, j \in \omega$. We shall call a subgroup K_j^i of G a $(\beta i, \alpha+j)$-subgroup of G if K_j^i is an $(\alpha+j)$-high subgroup of $p^{\beta i}G$.

We shall call a subgroup K_ω^i of G a $(\beta i, (\alpha, \omega))$-subgroup of G if $K_\omega^i = \bigcup_{j\in\omega} K_j^i$ for some chain $\{K_j^i\}_{j\in\omega}$ of $(\beta i, \alpha+j)$-subgroups of G.

THEOREM 2.7. *A reduced p-group G is ω-summable iff* $G = \langle\{K_\omega^i \mid i \in \omega\}\rangle$ *for some sequence* $\{K_\omega^i\}_{i\in\omega}$ *of* $(\omega i, (\omega, \omega))$*-subgroups of G such that* $p^\omega K_\omega^i \subset K_\omega^{i+1}$.

Proof. Clearly if $G = \langle\{K_\omega^i \mid i \in \omega\}\rangle$ we may take $G_i = \langle\{K_\omega^k \mid k < i-1\}, H\rangle$ where H is a high subgroup of K_ω^{i-1} such that $H \supset p^\omega K_\omega^{i-2}$ if $i > 1$. This follows from Lemma 1.10.

If $G = \bigcup_{i\in\omega^*} G_i$ with G_i ωi-high in G we may construct a sequence $\{K_\omega^i\}_{i\in\omega}$ as follows. Let K_i^0 be a chain of $(\omega\cdot 0, \omega+i)$-subgroups of G_2 with $K_0^0 = G_1$ and $K_\omega^0 = \bigcup_{i\in\omega} K_i$. Assuming that $K_\omega^n \subset G_{n+2}$ has been constructed let K_0^{n+1} be a $(\omega(n+1), \omega)$-subgroup of G_{n+2} containing $p^\omega K_\omega^n$ and $\{K_i^{n+1}\}_{i\in\omega}$ be a chain of $(\omega(n+1), \omega+i)$-subgroups of G_{n+3}. Let $K_\omega^{n+1} = \bigcup_{i\in\omega} K_i^{n+1}$. Then it follows easily from Lemma 1.10 that $G = \langle\{K_\omega^i \mid i \in \omega\}\rangle$.

It follows that if G is ω-summable then $G \cong (\sum_{i\in\omega} K_\omega^i)/R$ where K_ω^i is a group of Ulm type $\leqq 2$, $p^\omega K_\omega^i$ is isomorphic to a basic subgroup of K_ω^{i+1}, and R is a relation which identifies $p^\omega K_\omega^i$ with a particular basic subgroup of K_ω^{i+1}.

To construct K_ω^i, let $\sigma_i : p^{\omega i}G \to p^{\omega i}G/p^{\omega(i+1)}G = G_i$, the ith Ulm factor of G, be the natural homomorphism and $\{H_i\}_{i\in\omega}$ be a high chain of $p^{\omega i}G$. Let B_{i+1} be a basic subgroup of G_{i+1} and D_{i+1} a divisible hull of B_{i+1}. Then K_ω^i can be constructed (up to isomorphism) as the subdirect sum of G_i and D_{i+1} with high chain $\{\sigma_i(H_i)\}_{i\in\omega}$ and elements of infinite height B_{i+1}. (The set $\{z_{\alpha,i} \mid \alpha \in \Gamma_i, i \in \omega\}$ as defined in Remark 1.15 should be carefully chosen.)

THEOREM 2.8. *Let G and H be ω-summable groups and let* G_j *and* H_j *be the jth Ulm factors of G and H, respectively. A necessary and sufficient condition for G and H to be isomorphic is that there exist isomorphisms* $\varphi_j : G_j \to H_j$ *such that* $\{\varphi_j\}_{j\in\omega}$ *is compatible and the high chains in* (i) *of Definition* 2.2 *can be chosen such that they generate G and H, respectively.*

3. **Reduced p-primary abelian groups of countable Ulm type.** We shall need the following generalization of Ulm factor. If G is a reduced p-group and β is a limit ordinal such that $\beta = \omega$ or β is of type 2, then the αth *β-factor* of G is defined to be $p^{\beta\alpha}G/p^{\beta(\alpha+1)}G$. Note that the αth ω-factor is just the αth Ulm factor. Also if β is the Ulm type of G and $\{\alpha_i\}_{i\in\omega}$ is an admissible sequence converging to β, then we will define the ith *$\{\alpha_i\}$-factor* to be $p^{\omega\alpha_i}G/p^{\omega\alpha_{i+1}}G$.

In this section we will determine the structure of a reduced p-primary abelian group of countable Ulm type in terms of the structure of its various factors.

DEFINITION 3.1. Let G be a reduced p-group, let β be a (countable) limit ordinal of type 2, and let $\{\alpha_i\}_{i\in\omega}$ be an admissible sequence converging to β. Then G will be said to be *β-summable* if there exists a chain of subgroups $\{G_i\}_{i\in\omega^*}$ with G_i $\omega\alpha_i$-high in G such that $G = \bigcup_{i\in\omega^*} G_i$. A subgroup C of G is said to be a *β-summable subgroup* of G if there exists a chain of subgroups $\{G_i\}_{i\in\omega^*}$ of G with G_i $\omega\alpha_i$-high in G such that $C=\bigcup_{i\in\omega^*} G_i$.

Note that the definition of a β-summable group is independent of the choice of the admissible sequence converging to β.

The next theorem is an immediate generalization of Theorem 2.5. Let α and β be limit ordinals, $\{\alpha_i\}_{i\in\omega}$ a sequence of limit ordinals, and $j\in\omega$. Following the notation of the last section we call a subgroup K_j^i of G a $(\beta\alpha_i, \alpha+j)$-subgroup of G if K_j^i is an $(\alpha+j)$-high subgroup of $p^{\beta\alpha_i}G$ and a subgroup K_ω^i of G a $(\beta\alpha_i, (\alpha, \omega))$-subgroup of G if $K_\omega^i=\bigcup_{j\in\omega} K_j^i$ for some chain $\{K_j^i\}_{j\in\omega}$ of $(\beta\alpha_i, \alpha+j)$-subgroups of G.

Theorem 3.2. *Let β be a countable limit ordinal of type 2. A reduced p-group G is β-summable iff there exists an admissible sequence $\{\alpha_i\}_{i\in\omega}$ of limit ordinals converging to β and a sequence $\{K_\omega^i\}_{i\in\omega}$ of $(\omega\alpha_i, (\omega(\alpha_{i+1}-\alpha_i), \omega))$-subgroups of G with $p^{\omega(\alpha_{i+1}-\alpha_i)}K_\omega^i\subset K_\omega^{i+1}$ such that $G=\langle\{K_\omega^i \mid i\in\omega\}\rangle$.*

The proof is similar to that of Theorem 2.5.

It follows that, if G is β-summable, then $G\cong(\sum_{i\in\omega} K_\omega^i)/R$ where K_ω^i is a group of type $\leqq(\alpha_{i+1}-\alpha_i)+1$ and R is a relation which identifies $p^\omega K_\omega^i$ with a particular basic subgroup of K_ω^{i+1}.

To construct K_ω^i, let $\sigma_i\colon p^{\omega\alpha_i}G\to p^{\omega\alpha_i}G/p^{\omega(\alpha_{i+1}-\alpha_i)}G=G_i$, the ith $\{\alpha_i\}$-factor, be the natural homomorphism and $\{H_i\}_{i\in\omega}$ be a chain of subgroups of $p^{\omega\alpha_i}G$ such that H_i is $(\omega(\alpha_{i+1}-\alpha_i)+i)$-high in $p^{\omega\alpha_i}G$. Let B_{i+1} be a basic subgroup of G_{i+1} and D_{i+1} a divisible hull of B_{i+1}. Then K_ω^i can be constructed (up to isomorphism) as the subdirect sum of G_i and D_{i+1} with $\omega(\alpha_{i+1}-\alpha_i)$-high chain $\{\sigma_i(H_i)\}_{i\in\omega}$ and $p^{\omega(\alpha_{i+1}-\alpha_i)}K_\omega^i=B_{i+1}$. (Again the set $\{z_{\alpha,i} \mid \alpha\in\Gamma_i, i\in\omega\}$ as defined in Remark 1.15 should be carefully chosen.) (That $\sigma_i(H_i)$ is isotype and dense in the $(\alpha_{i+1}-\alpha_i)$-topology follows from Lemma 1.1 and results of [8]. The proofs are similar to those in [3].)

Remark 3.3. Let β be a countable ordinal $>\omega$ and G a reduced p-group of Ulm type β. If β is not a limit ordinal or β is of type 1 then there exists a limit ordinal $\gamma<\beta$ such that γ is of type 2 and $\gamma\omega>\beta$. In this case a theorem similar to Theorem 2.1 can be stated with the Ulm factors replaced by the $\omega\gamma$-factors, etc. If β is a limit ordinal of type 2, let $\{\alpha_i\}$ be an admissible sequence converging to β. Then a theorem similar to Theorem 2.8 can be stated for β-summable p-groups with the Ulm factors replaced by the $\{\alpha_i\}$-factors, etc.

Lemma 3.4. *Let G be a reduced p-group of Ulm type β such that β is a countable limit ordinal of type 2 (or $\beta=\omega$). Let C be a β-summable subgroup of G. Then C is a dense subgroup of G with respect to the β-topology on G.*

Proof. Let $x\in G$ such that $O(x)=p^k$ and let $\tilde{\alpha}=\{\alpha_i\}_{i\in\omega}$ be a $(\beta-U)$-sequence. We will show that $(x+G\{\tilde{\alpha}\})\cap C\neq\varnothing$. Let γ be the least limit ordinal greater than or equal to α_k. Then $(p^\gamma G)[p^k]\subset G\{\tilde{\alpha}\}$. (Note that $\gamma<\omega\beta$ since, otherwise, $\omega\beta=\alpha_i+\omega$.) Let K be a $(\gamma+k)$-high subgroup of C, and let M be a complementary summand of a maximal p^k-bounded summand of $p^\gamma G$. Then $G[p^k]=K[p^k] \oplus M[p^k]$ by Lemmas 1.1 and 1.10. Thus $x=y+z$ where $y \in K[p^k]$ and $z \in M[p^k]$. Thus $x+G\{\tilde{\alpha}\}=y+G\{\tilde{\alpha}\}$ and since $G\{\tilde{\alpha}\} \cap C \neq \varnothing$ we must have $(x+G\{\tilde{\alpha}\}) \cap C \neq \varnothing$.

LEMMA 3.5. *Let C be a p-group of length β, a countable limit ordinal of type 2 (or $\beta=\omega$). Let $\bar{C}$ be the completion of C with respect to the β-topology. Then C is isotype in $\bar{C}$.*

Proof. Suppose $p^\alpha C=C \cap p^\alpha \bar{C}$ for all $\alpha<\gamma$. If γ is a limit ordinal then it is easily shown that $p^\gamma C=C \cap p^\gamma \bar{C}$. If γ is not a limit ordinal then $\gamma=\alpha+1$ for some ordinal α. Let $x \in C \cap p^\gamma \bar{C}$. Let $y \in p^\alpha \bar{C}$ such that $py=x$. There exists a sequence $\{y_i\}_{i\in\omega}$ in C converging to y. For sufficiently large i, $y_i-y \in p^\alpha \bar{C}$ and hence $y_i \in p^\alpha \bar{C} \cap C$ for these i. Therefore, py_i-x, $py_i \in p^\gamma C$ for large i; thus $x \in p^\gamma C$. The proof follows.

THEOREM 3.6. *Let G be a reduced p-group and let β be a countable limit ordinal of type 2 (or $\beta=\omega$). Let C be a β-summable subgroup of G. Then $G/p^{\omega\beta}G$ is isomorphic to an isotype subgroup V between C and $\bar{C}$, the completion of C with respect to the β-topology.*

Proof. That $G/p^{\omega\beta}G$ is isomorphic to an isotype subgroup between C and $\bar{C}$ follows from Lemmas 3.2 and 3.5 above and Proposition 5, p. 246 in [1].

THEOREM 3.7. *Let β be an ordinal of type 2 (or $\beta=\omega$) and let G be a reduced p-group of Ulm type $\geqq\beta$. Let S be a subsocle of $G[p]$ dense with respect to the β-topology on G. Let L be a subgroup of G such that $L[p]\subset S$. Let H be a maximal subgroup of G containing L and supported by S. Then H is $p^{\omega\beta}$-pure(1) in G.*

Proof. Let $\alpha<\omega\beta$. Since S is dense in $G[p]$ we may write $G[p]=S \oplus P$ such that $P\subset p^\alpha G$. (If not, there exists $x \in G[p]-S$ such that the height of x is $<\alpha$ and $x+(p^\alpha G)[p] \cap S=0$ contradicting the density of S.) By Theorem 2 of [8] we have that H is $p^{\alpha+1}$-pure in G. Thus it follows easily that H is $p^{\omega\beta}$-pure in G.

COROLLARY 3.8. *If the type of G is $\leqq\beta$ then H is isotype in G.*

COROLLARY 3.9. *If G is a group of type $\leqq\beta$ then G and H have the same (up to isomorphism) Ulm factors.*

We will now proceed to give a general construction for groups of countable Ulm type. I.e., if τ is a countable ordinal, $\aleph$ is a cardinal, and $\{G_\alpha\}_{\alpha\in\tau}$ is a sequence of p-groups without elements of infinite height satisfying

(i) $\sum_{0\leqq\alpha<\tau} |G_\alpha| \leqq \aleph \leqq \prod_{0\leqq\alpha<\min(\omega,\tau)} |G_\alpha|$,

(ii) $\sum_{\beta\leqq\alpha<\tau} |G_\alpha| \leqq |G_\beta|^{\aleph_0}$ for all $0\leqq\beta<\tau$,

(iii) $r(B_{\alpha+1})\leqq \text{fin } r(G_\alpha)$ for all $1\leqq\alpha+1<\tau$,

where $B_{\alpha+1}$ is a basic subgroup of $G_{\alpha+1}$, we will construct all reduced p-groups G satisfying

(a) $|G|=\aleph$,

(1) The terminology currently used is weakly $p^{\omega\beta}$-pure.

(b) $\tau(G)=\tau$, and

(c) the Ulm sequence of G is $\{G_\alpha\}_{\alpha\in\tau}$.

Let $\{G_\alpha\}_{\alpha\in\omega}$ be a sequence of p-groups with no elements of infinite height satisfying conditions (i) and (iii). For each $\alpha\in\omega$ let B_α be a basic subgroup of G_α such that $r(G_\alpha/B_\alpha)\geqq r(B_{\alpha+1})$. Having chosen the B_α's, let H_α be a pure subgroup of G_α containing B_α such that $r(G_\alpha/H_\alpha)=r(B_{\alpha+1})$ and let $D_{\alpha+1}$ be a divisible hull of $B_{\alpha+1}$. Let K_α be a subdirect sum of G_α and $B_{\alpha+1}$ with high subgroup H_α and elements of infinite height $B_{\alpha+1}$. Let $C=(\sum_{\alpha\in\omega} K_\alpha)/R$ where R is the relation that identifies $p^\omega(K_\alpha)$ with the basic subgroup $B_{\alpha+1}$ in $K_{\alpha+1}$. Let $\bar{C}$ be the completion of C with respect to the ω-topology. Let S be a subsocle of $\bar{C}[p]$ containing $C[p]$, with $|S|=\aleph$, and let G be an isotype subgroup of $\bar{C}$ supporting S. Then G is a reduced p-group satisfying conditions (a)–(c). Using the results of §§1 and 2 we note that all reduced p-groups of type $\leqq\omega$ can be constructed in this manner.

Next let β be a countable ordinal $>\omega$ and suppose that all p-groups of type $<\beta$ have been constructed, and, moreover, if $\gamma<\beta$ and $\{G_\alpha\}_{\alpha\in\gamma}$ is a sequence of p-groups without elements of infinite height satisfying conditions (i)–(iii), then a group G has been constructed satisfying conditions (a)–(c) (one may take $|G|<\aleph$ where appropriate).

Let $\{G_\alpha\}_{\alpha\in\beta}$ be a sequence of p-groups with no elements of infinite height satisfying conditions (i)–(iii).

If β is not a limit ordinal or if β is an ordinal of type 1, let ρ be a limit ordinal of type 2 (or $\rho=\omega$) such that $\rho<\beta$ and $\rho\omega>\beta$ (the least limit ordinal ρ such that $\rho\omega\geqq\beta$ will do). Let n be the least positive integer such that $\rho n\geqq\beta$. Let C_i for $i=0,\dots,n-2$ be a ρ-summable reduced p-group such that the Ulm sequence of C_i is $\{G_\alpha\}_{\rho i\leqq\alpha<\rho(i+1)}$ and $|C_i|\leqq\aleph$, and let G_{n-1} be a reduced p-group such that the Ulm sequence of G_{n-1} is $\{G_\alpha\}_{\rho(n-1)\leqq\alpha<\beta}$ and $|G_{n-1}|\leqq\aleph$. Next let $\bar{C}_i$ be the completion of C_i with respect to the ρ-topology and let G_i be an isotype subgroup of $\bar{C}_i$ containing C_i such that $|G_i|\leqq\aleph$ and $r(G_i/C_i)\geqq r(B_{i+1})$, B_{i+1} a basic subgroup of C_{i+1} (or G_{n-1} if $i=n-2$). In the choice of the G_i's we must choose at least one with cardinality $\aleph$. Let H_i be an isotype subgroup of G_i containing C_i such that $r(G_i/H_i)=r(B_{i+1})$, $i=0,\dots,n-2$. Let K_i be the subdirect sum of G_i and B_{i+1} with ρ-high subgroup H_i and $p^{\omega\rho}K_i\cong B_{i+1}$ for $i=0,\dots,n-2$, and let $K_{n-1}=G_{n-1}$. Let $G=(\sum_{i=0}^{n-1} K_i)/R$ where R is the relation that identifies $p^{\omega\rho}K_i$ with B_{i+1} the basic subgroup of G_{i+1}. Then G satisfies conditions (a)–(c).

If β is an ordinal of type 2, let $\{\alpha_i\}$ be an admissible sequence converging to β. Let $\{C_i\}_{i\in\omega}$ be a sequence of reduced p-groups such that C_i is $(\alpha_{i+1}-\alpha_i)$-summable, the Ulm sequence of C_i is $\{G_\alpha\}_{\alpha_i\leqq\alpha<\alpha_{i+1}}$ and $|C_i|\leqq\aleph$. For each $i\in\omega$ let B_i be a basic subgroup of C_i, let $\bar{C}_i$ be the completion of C_i with respect to the $(\alpha_{i+1}-\alpha_i)$-topology, and let G_i be an isotype subgroup of $\bar{C}_i$ containing C_i such that $r(G_i/C_i)\geqq r(B_{i+1})$ and $|G_i|\leqq\aleph$. For each $i\in\omega$ let H_i be an isotype subgroup of G_i containing C_i such that $r(G_i/H_i)=r(B_{i+1})$ and let D_{i+1} be a divisible hull of B_{i+1}. Let K_i be a subdirect sum of G_i and B_{i+1} with $(\alpha_{i+1}-\alpha_i)$-high subgroup H_i and

$p^{\omega(\alpha_{i+1}-\alpha_i)}G_i \cong B_{i+1}$. Let $C=(\sum G_i)/R$ where R is the relation that identifies the $p^{\omega(\alpha_{i+1}-\alpha_i)}G_i$'s with B_{i+1}, the basic subgroup of G_{i+1}. Let $\bar{C}$ be the completion of C with respect to the β-topology, and let G be an isotype subgroup of $\bar{G}$ containing C such that $|G|=\aleph$. Then G satisfies conditions (a)–(c) and we see from §3 that every group of length β is obtainable in this manner.

4. **Examples.** The following facts will be needed to justify some of the statements in the construction of the examples.

Lemma 4.1. *Let V be a vector space over a field F with infinite dimension $r(V)$. Let $V=M \oplus N$ such that $M\neq 0$ and $r(N)=m$ an infinite cardinal number. Then there exists at least 2^m (distinct) complementary summands of M in V.*

Proof. Let $N=\sum_{\alpha\in\beta}\langle x_\alpha\rangle$ with $|\beta|=m$. Let $0\neq x\in M$. Let

$$\mathscr{S} = \{\{z_\alpha \mid \alpha\in\beta\} \mid z_\alpha \in \{x_\alpha, x_\alpha+x\} \text{ for each } \alpha\in\beta\}.$$

Then $|\mathscr{S}|=2^m$; if $L, L'\in\mathscr{S}$ such that $L\neq L'$ then $\langle L\rangle\neq\langle L'\rangle$; and for each $L\in\mathscr{S}$, $V=M\oplus\sum_{x\in L}\langle x\rangle$.

Corollary 4.2. *If G is an unbounded reduced p-group with countable basic subgroup such that $|G[p]|=m$, then there exists 2^m dense subsocles Q of $G[p]$ such that $G[p]/Q$ has rank one. In fact there exists 2^m such subsocles containing the socle of the same basic subgroup.*

Proposition 4.3. *Let $\bar{B}$ be a closed p-group with unbounded countable basic subgroup B. Then there exists a set $\mathscr{G}$ of 2^c nonisomorphic pure subgroups between B and $\bar{B}$ such that*

(1) *if $G\in\mathscr{G}$ then $\bar{B}/G\cong Z(p^\infty)$;*

(2) *if $G, H\in\mathscr{G}$, $G\neq H$, then $G[p]\neq H[p]$;*

(3) *if $G, H\in\mathscr{G}$, $G\neq H$, there does not exist an automorphism φ of $\bar{B}$ such that $\varphi(G[p])=H[p]$.*

Proof. Since there are at most c automorphisms of $\bar{B}$ (see [5]) the theorem follows from Corollary 4.2.

Example 1. Let G and H be reduced p-groups with high subgroups K and L, respectively, such that $G/p^\omega G\cong H/p^\omega H$ under an isomorphism φ, $p^\omega G\cong p^\omega H$, and if $\sigma\colon G\to G/p^\omega G$ and $\rho\colon H\to H/p^\omega H$ are natural then $\varphi(\sigma(H))=\rho(L)$. If $\rho^\omega G$ is a direct sum of cyclic groups all of the same order, it follows by Theorem 1 that G and H are isomorphic. The following example will show that we cannot remove the restriction that $p^\omega G$ be a direct sum of cyclic groups all of the same order and have G and H necessarily isomorphic. In fact, if $p^\omega G\cong p^\omega H\cong\langle x_1\rangle\oplus\langle x_2\rangle$ such that $O(x_i)=p^i$ for $i=1, 2$ then there exists such nonisomorphic groups G and H.

Let $B=\sum_{i=1}^\infty\langle b_i\rangle$ such that $O(b_i)=p^i$. Let $\bar{B}$ be the torsion completion of B. Let Q and R be subsocles of $\bar{B}$ containing $B[p]$ such that

(1) $\bar{B}[p]/Q\cong\bar{B}[p]/R\cong Z(p)$ and

(2) there exists no automorphism φ of $\bar{B}$ such that $\varphi(Q)=R$.

Let L be a pure subgroup of $\bar{B}$ such that $L[p]=R \cap Q$. Let K_Q and K_R be pure subgroups of $\bar{B}$ containing L such that $K_Q[p]=Q$ and $K_R[p]=R$. Let $M=\langle x_1\rangle \oplus \langle x_2\rangle$ such that $O(x_i)=p^i$, $i=1, 2$. Let D_i be a divisible hull of $\langle x_i\rangle$, $i=1, 2$, and let $D=D_1 \oplus D_2$. Now $K_Q[p]=L[p] \oplus \langle z\rangle$, $K_R[p]=L[p] \oplus \langle w\rangle$, and $\bar{B}[p]=L[p] \oplus \langle z\rangle \oplus \langle w\rangle$. Let $d_i \in D_i$, $i=1, 2$, such that $pd_i=x_i$. Define isomorphisms φ_1: $\bar{B}/L \to D/M$ such that $\varphi_1(z+L)=d_1+M$, $\varphi_1(w+L)=d_2+M$, and $\varphi_1(K_Q/L)=D_1$ and φ_2: $\bar{B}/L \to D/M$ such that $\varphi_2(z+L)=d_2+M$, $\varphi_2(\omega+L)=d_1+M$, and $\varphi_2(K_R/L)=D_1$.

Let α: $\bar{B} \to \bar{B}/L$ and β: $D \to D/M$ be natural. Define subgroups G and H of $\bar{B} \oplus D$ as $G=\{(b, d) \mid b \in \bar{B}, d \in D \text{ and } \varphi_1\alpha(b)=\beta(d)\}$ and $H=\{(b, d) \mid b \in \bar{B}, d \in D \text{ and } \varphi_2\alpha(b)=\beta(d)\}$. Thus $p^\omega G=\{0\} \oplus M \cong p^\omega H \cong M$, $G/p^\omega G \cong H/p^\omega H \cong \bar{B}$, $L \oplus \{0\}$ is a high subgroup of G and H, and any isomorphism between the embeddings of high subgroups of G and H in $G/p^\omega G$ and $H/p^\omega H$, respectively, extends to an isomorphism between $G/p^\omega G$ and $H/p^\omega H$ (see [11]).

Suppose that φ is an isomorphism from G onto H. Note that

$$\hat{K}_Q = \{(k, d) \in G \mid k \in K_Q \text{ and } d \in D_1\}$$

is $\langle x_2\rangle$-high in G; $\sigma(\hat{K}_Q)=K_Q$ where σ: $G \to G/p^\omega G$ is natural;

$$\hat{K}_R = \{(k, d) \in H \mid k \in K_R \text{ and } d \in D_1\}$$

is $\langle x_2\rangle$-high in H; and $\gamma(\hat{K}_R)=K_R$ where γ: $H \to H/p^\omega H$ is natural. Now $\varphi(\hat{K}_Q)$ is $\langle x_2\rangle$-high in L and $\gamma(\varphi(\hat{K}_Q))[p]=\gamma(\hat{K}_R)[p]$ (by Lemma 4 in [8]). Hence φ induces an automorphism of $\bar{B}$ taking $K_Q[p]$ onto $K_R[p]$ which is a contradiction. Thus G and L are not isomorphic.

Example 2. This example shows that the connection between the Ulm factors described in conditions (b) and (c) of Definition 2.2 must be given.

Let G be a reduced p-group such that $G/p^\omega G \cong p^\omega G \cong \bar{B}$ where B is a countable unbounded direct sum of cyclic groups. Let $\{G_i\}_{i\in\omega}$ be a high chain in G. Let $G_\omega=\bigcup_{i=0}^{\infty} G_i$. Let $x \in p^\omega G[p]-p^\omega G_\omega[p]$. Let H be an isotype subgroup of G containing G_ω such that $H[p]=G_\omega[p] \oplus \langle x\rangle$. Now by [5] there exists 2^c non-isomorphic pure subgroups K of H such that $K[p]=G_\omega[p]$ and $p^\omega G_\omega \subset K$. Note that $K/p^\omega K \cong G_\omega/p^\omega G_\omega \cong H/p^\omega H$. Thus some K is not isomorphic to H. But if $\{K_i\}_{i\in\omega}$ is a high chain in K, then K_i is $(\omega+i)$-high in H. If σ: $H \to H/p^\omega H$ and σ': $K \to K/p^\omega K$ are natural, then φ: $K/p^\omega K \to H/p^\omega H$: $k+p^\omega K \to k+p^\omega H$ is an isomorphism such that $\varphi(\sigma'(K_i))=\sigma(K_i)$. Also $p^\omega K$ and $p^\omega H$ are isomorphic. But K and H are not isomorphic. Thus the connection must be prescribed.

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